A remark on decay rates of solutions for a system of quadratic nonlinear Schrödinger equations in 2D

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Abstract: We consider the initial value problem for a three-component system of quadratic nonlinear Schrödinger equations with mass resonance in two space dimensions. Under a suitable condition on the coefficients of the nonlinearity, we will show that the solution decays strictly faster than $O(t^{-1})$ as $t \to +\infty$ in L^∞ by providing with an enhanced decay estimate of order $O((t \log t)^{-1})$. Differently from the previous works, our approach does not rely on the explicit form of the asymptotic profile of the solution at all.

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1 Introduction and the main result

This paper is intended to be a sequel of the papers [8] and [9] by one of the authors, which are concerned with decay property of solutions to the initial value problem for a class of nonlinear Schrödinger systems. The model system which we focus on here is

$$\begin{cases} i\partial_t u_1 + \dfrac{1}{2m_1}\Delta u_1 = \lambda_1 |u_1| u_1 + \mu_1 \overline{u_2} u_3, \\ i\partial_t u_2 + \dfrac{1}{2m_2}\Delta u_2 = \lambda_2 |u_2| u_2 + \mu_2 \overline{u_1} u_3, \qquad t > 0, \quad x \in \mathbb{R}^2 \\ i\partial_t u_3 + \dfrac{1}{2m_3}\Delta u_3 = \lambda_3 |u_3| u_3 + \mu_3 u_1 u_2, \end{cases} \tag{1.1}$$

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with

$$u_j(0,x) = \varphi_j(x), \qquad x \in \mathbb{R}^2, \quad j = 1,2,3 \tag{1.2}$$

(see, e.g., [1], [2], [4] for the physical background of this system), where $m_1, m_2, m_3 \in \mathbb{R}\setminus\{0\}$, $\lambda_1, \lambda_2, \lambda_3$, $\mu_1, \mu_2, \mu_3 \in \mathbb{C} \setminus \{0\}$ are constants, $\varphi = (\varphi_j(x))_{j=1,2,3}$ is a prescribed $\mathbb{C}^3$-valued function, and $u = (u_j(t,x))_{j=1,2,3}$ is a $\mathbb{C}^3$-valued unknown function. As usual, $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, and Δ is the Laplacian in x-variables.

By a minor modification of the method of [8] and [9], we can show the following basic L^∞-decay result. Here and hereafter, we denote by $H^{s,\sigma}(\mathbb{R}^2)$ the weighted Sobolev spaces, i.e.,

$$H^{s,\sigma}(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) \mid (1+|x|^2)^{\sigma/2}(1-\Delta)^{s/2} f \in L^2(\mathbb{R}^2)\}$$

equipped with the norm $\|f\|_{H^{s,\sigma}} = \left\|(1+|x|^2)^{\sigma/2}(1-\Delta)^{s/2} f\right\|_{L^2}$.

Proposition 1.1. *Assume*

$$m_1 + m_2 = m_3, \tag{1.3}$$

$$\operatorname{Im} \lambda_j \le 0 \quad \textit{for } j = 1,2,3, \tag{1.4}$$

$$\kappa_1 \mu_1 + \kappa_2 \mu_2 = \kappa_3 \overline{\mu_3} \quad \textit{with some } \kappa_1,\ \kappa_2,\ \kappa_3 > 0, \tag{1.5}$$

and $\varphi = (\varphi_j)_{j=1,2,3} \in H^{s,0}(\mathbb{R}^2) \cap H^{0,s}(\mathbb{R}^2)$ *with* $1 < s < 2$. *Then there exists a positive constant* $\varepsilon_0 > 0$ *such that the initial value problem* (1.1)–(1.2) *admits a unique global solution*

$$u \in C\big([0,\infty); H^{s,0}(\mathbb{R}^2) \cap H^{0,s}(\mathbb{R}^2)\big),$$

provided that $\|\varphi\|_{H^{s,0}} + \|\varphi\|_{H^{0,s}} \le \varepsilon_0$. *Moreover, there exists a positive constant* C_0 *such that*

$$\|u(t,\cdot)\|_{L^\infty} \le \frac{C_0}{1+t} \tag{1.6}$$

for all $t \ge 0$.

In fact, $\lambda_j |u_j| u_j$ was not included in the nonlinearity considered in [8] and [9]. However, as verified in Section 3, it is straightforward to modify the proof under the assumption (1.4) on the coefficient λ_j. In view of the conservation law

$$\sum_{j=1}^{3} \kappa_j \left(\|u_j(t,\cdot)\|_{L^2}^2 - 2 \operatorname{Im} \lambda_j \int_0^t \|u_j(\tau,\cdot)\|_{L^3}^3 d\tau \right) = \sum_{j=1}^{3} \kappa_j \|u_j(0,\cdot)\|_{L^2}^2,$$

it may be natural to regard (1.4) as a kind of dissipativeness condition. This will lead us to the following question: *Is the decay rate* $O(t^{-1})$ *in* (1.6) *enhanced if the inequalities in* (1.4) *are strict?* To the authors' knowledge, there is no previous result which answers this

question except the case where the system is decoupled, i.e., $\mu_1 = \mu_2 = \mu_3 = 0$. In the decoupled case, the problem is reduced to the single equation

$$i\partial_t v + \frac{1}{2}\Delta v = \lambda |v| v, \quad t > 0, \quad x \in \mathbb{R}^2 \tag{1.7}$$

with $\operatorname{Im}\lambda < 0$, and the solution $v(t,x)$ decays like $O((t\log t)^{-1})$ in the sense of L^∞ for small initial data, according to [10] (see also [5], [6] and [11]). However, the proof of [10] is not applicable to the present setting because it heavily depends on the facts that the solution $v(t,x)$ to (1.7) is well-approximated by

$$\frac{e^{i\frac{|x|^2}{2t}}}{it}\alpha\left(t, \frac{x}{t}\right),$$

where $\alpha(t,\xi)$ solves

$$i\partial_t \alpha = \frac{\lambda}{t}|\alpha|\alpha + o(t^{-1}),$$

and that $\alpha(t,\xi)$ behaves like

$$\frac{\alpha(1,\xi)}{1 - |\alpha(1,\xi)| \operatorname{Im}\lambda \log t} \exp\left(i \frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda} \log(1 - |\alpha(1,\xi)| \operatorname{Im}\lambda \log t) \right)$$

as $t \to +\infty$ if $\operatorname{Im}\lambda < 0$. On the other hand, the corresponding reduced ODE system for (1.1) is

$$\begin{cases} i\partial_t \alpha_1 = \dfrac{1}{t}\left(\lambda_1 |\alpha_1| \alpha_1 + \mu_1 \overline{\alpha_2} \alpha_3\right) + o(t^{-1}), \\ i\partial_t \alpha_2 = \dfrac{1}{t}\left(\lambda_2 |\alpha_2| \alpha_2 + \mu_2 \overline{\alpha_1} \alpha_3\right) + o(t^{-1}), \\ i\partial_t \alpha_3 = \dfrac{1}{t}\left(\lambda_3 |\alpha_3| \alpha_3 + \mu_3 \alpha_1 \alpha_2\right) + o(t^{-1}), \end{cases} \tag{1.8}$$

which is much more complicated than the single case.

The aim of this paper is to give an affirmative answer to the above question by providing with an enhanced decay estimate of order $O((t\log t)^{-1})$. The novelty of the present approach is that it does not rely on the explicit form of the asymptotic profile of the solution at all. The main result is as follows.

Theorem 1.1. *Suppose that the assumptions of Proposition 1.1 are fulfilled. Let u be the solution to* (1.1)–(1.2) *whose existence is guaranteed by Proposition 1.1. If*

$$\operatorname{Im}\lambda_j < 0 \quad \textit{for } j = 1,2,3 \tag{1.9}$$

is satisfied, then there exists a positive constant C_1 such that

$$\|u(t,\cdot)\|_{L^\infty} \le \frac{C_1}{(1+t)\log(2+t)}$$

for all $t \ge 0$.

2 Preliminaries

In this section, we introduce several notations and lemmas which will be needed in the subsequent sections. In what follows, we denote by $\langle \cdot, \cdot \rangle_{\mathbb{C}^3}$ the standard scalar product in $\mathbb{C}^3$, i.e.,

$$\langle z, w \rangle_{\mathbb{C}^3} = \sum_{j=1}^{3} z_j \overline{w_j}$$

for $z = (z_j)_{j=1,2,3}$, $w = (w_j)_{j=1,2,3} \in \mathbb{C}^3$. We also write $|z|_{\mathbb{C}^3} = \sqrt{\langle z, z \rangle_{\mathbb{C}^3}}$, as usual.

First we rewrite (1.1) in the abstract form: Put $\Lambda u = (\frac{1}{2m_j} \Delta u_j)_{j=1,2,3}$ so that

$$i\partial_t u + \Lambda u = F(u),$$

where $F : \mathbb{C}^3 \ni z = (z_j)_{j=1,2,3} \mapsto (F_j(z))_{j=1,2,3} \in \mathbb{C}^3$ is defined by

$$\begin{aligned} F_1(z) &= \lambda_1 |z_1| z_1 + \mu_1 \overline{z_2} z_3, \\ F_2(z) &= \lambda_2 |z_2| z_2 + \mu_2 \overline{z_1} z_3, \\ F_3(z) &= \lambda_3 |z_3| z_3 + \mu_3 z_1 z_2. \end{aligned}$$

Then the assumptions (1.3), (1.4) and (1.5) on the coefficients can be interpreted as follows:

- We define $\mathcal{E}_\theta : \mathbb{C}^3 \to \mathbb{C}^3$ by $\mathcal{E}_\theta z = (e^{i m_j \theta} z_j)_{j=1,2,3}$ for $z = (z_j)_{j=1,2,3} \in \mathbb{C}^3$ and $\theta \in \mathbb{R}$. Then we have

  $$\mathcal{E}_\theta F(z) = F(\mathcal{E}_\theta z) \tag{2.1}$$

  for all $\theta \in \mathbb{R}$ and $z \in \mathbb{C}^3$, provided that (1.3) is satisfied.

- With κ_1, κ_2, κ_3 appearing in (1.5), we put

  $$A = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix}.$$

  Then, by (1.5), we have

  $$\langle F(z), Az \rangle_{\mathbb{C}^3} = \sum_{j=1}^{3} \kappa_j \lambda_j |z_j|^3 + 2\kappa_3 \operatorname{Re}\Big(\overline{\mu_3 z_1 z_2} z_3 \Big),$$

  whence (1.4) implies

  $$\operatorname{Im} \langle F(z), Az \rangle_{\mathbb{C}^3} = \sum_{j=1}^{3} \kappa_j \operatorname{Im} \lambda_j |z_j|^3 \leq 0$$

for all $z \in \mathbb{C}^3$. In particular, if (1.9) is satisfied, we can take positive constants C^* and C_* such that

$$-C^*\nu_A(z)^3 \leq \mathrm{Im}\langle F(z), Az\rangle_{\mathbb{C}^3} \leq -C_*\nu_A(z)^3 \tag{2.2}$$

for $z \in \mathbb{C}^3$, where $\nu_A(z) = \sqrt{\langle z, Az\rangle_{\mathbb{C}^3}}$. Note that

$$\sqrt{\kappa_*}|z|_{\mathbb{C}^3} \leq \nu_A(z) \leq \sqrt{\kappa^*}|z|_{\mathbb{C}^3}$$

for $z \in \mathbb{C}^3$, where $\kappa_* = \min\{\kappa_1, \kappa_2, \kappa_3\}$ and $\kappa^* = \max\{\kappa_1, \kappa_2, \kappa_3\}$.

Next we set $\mathcal{U}(t) = \exp(it\Lambda)$, i.e.,

$$\mathcal{U}(t)\phi = \left(\exp\left(\frac{it}{2m_j}\Delta\right)\phi_j\right)_{j=1,2,3}$$

for a $\mathbb{C}^3$-valued smooth function $\phi = (\phi_j)_{j=1,2,3}$. Also we set

$$\mathcal{G}\phi(\xi) = \left(-im_j\hat{\phi}_j(m_j\xi)\right)_{j=1,2,3},$$

where $\hat{f}$ denotes the Fourier transform of f, i.e.,

$$\hat{f}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-iy\cdot\xi}f(y)dy.$$

Now we summarize several useful lemmas. Since they are essentially not new or rather standard, we will give only an outline of the proof in the Appendix.

Lemma 2.1. *Let $s \in [0, 2)$. There exists a constant C such that*

$$\|F(\phi)\|_{H^{s,0}} \leq C\|\phi\|_{L^\infty}\|\phi\|_{H^{s,0}}.$$

Lemma 2.2. *Let $\gamma \in [0, 1]$ and $s > 1 + 2\gamma$. There exists a constant C such that*

$$\left|\|\phi\|_{L^\infty} - t^{-1}\|\mathcal{G}\mathcal{U}(-t)\phi\|_{L^\infty}\right| \leq Ct^{-1-\gamma}\|\mathcal{U}(-t)\phi\|_{H^{0,s}}$$

for $t \geq 1$.

Lemma 2.3. *Let $s \in [0, 2)$. Assume that (1.3) is satisfied. Then there exists a constant C such that*

$$\|\mathcal{U}(-t)F(\phi)\|_{H^{0,s}} \leq C\|\phi\|_{L^\infty}\|\mathcal{U}(-t)\phi\|_{H^{0,s}}.$$

Lemma 2.4. *Let $\gamma \in [0, 1/2)$ and $1 + 2\gamma < s < 2$. Assume that (1.3) is satisfied. Then there exists a constant C such that*

$$\left\|\mathcal{G}\mathcal{U}(-t)F(\phi) - \frac{1}{t}F\big(\mathcal{G}\mathcal{U}(-t)\phi\big)\right\|_{L^\infty} \leq Ct^{-1-\gamma}\|\mathcal{U}(-t)\phi\|_{H^{0,s}}^2$$

for $t \geq 1$.

3 A priori estimate

The argument of this section is almost the same as those of the previous works [4], [8], [9], [10]. Let $u(t)$ be the solution to (1.1)–(1.2) in the interval $[0,T]$. We define

$$\|u\|_{X_T} = \sup_{t\in[0,T]}\Big\{(1+t)\|u(t,\cdot)\|_{L^\infty} + (1+t)^{-\gamma/3}\big(\|u(t,\cdot)\|_{H^{s,0}} + \|\mathcal{U}(-t)u(t,\cdot)\|_{H^{0,s}}\big)\Big\},$$

where $1 < s < 2$ and $0 < \gamma < \frac{s-1}{2}$. We also set $\varepsilon = \|\varphi\|_{H^{s,0}} + \|\varphi\|_{H^{0,s}}$.

Lemma 3.1. *There exists a constant C, independent of T, such that*

$$\|u\|_{X_T} \le C\varepsilon + C\|u\|_{X_T}^2.$$

Remark 3.1. The above estimate implies that there exists a constant $C_0 > 0$, which does not depend on T, such that

$$\|u\|_{X_T} \le C_0\varepsilon \tag{3.1}$$

if we choose ε sufficiently small. Proposition 1.1 is an immediate consequence of this *a priori* bound and the standard local existence theorem.

Proof of Lemma 3.1. In what follows, we denote several positive constants by the same letter C, which may vary from one line to another. First we consider the estimates for $\|u(t,\cdot)\|_{H^{s,0}}$ and $\|\mathcal{U}(-t)u(t,\cdot)\|_{H^{0,s}}$. By the standard energy inequality combined with Lemmas 2.1 and 2.3, we have

$$\begin{aligned}
\|u(t,\cdot)\|_{H^{s,0}} \le& \|\varphi\|_{H^{s,0}} + \int_0^t \|F(u(\tau,\cdot))\|_{H^{s,0}}d\tau \\
\le& C\varepsilon + C\int_0^t \|u(\tau,\cdot)\|_{L^\infty}\|u(\tau,\cdot)\|_{H^{s,0}}d\tau \\
\le& C\varepsilon + C\|u\|_{X_T}^2\int_0^t \frac{d\tau}{(1+\tau)^{1-\gamma/3}} \\
\le& C\varepsilon + C\|u\|_{X_T}^2(1+t)^{\gamma/3}
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
\|\mathcal{U}(-t)u(t,\cdot)\|_{H^{0,s}} \le& \|\varphi\|_{H^{0,s}} + \int_0^t \|\mathcal{U}(-\tau)F(u(\tau,\cdot))\|_{H^{0,s}}d\tau \\
\le& C\varepsilon + C\int_0^t \|u(\tau,\cdot)\|_{L^\infty}\|\mathcal{U}(-\tau)u(\tau,\cdot)\|_{H^{0,s}}d\tau \\
\le& C\varepsilon + C\|u\|_{X_T}^2\int_0^t \frac{d\tau}{(1+\tau)^{1-\gamma/3}} \\
\le& C\varepsilon + C\|u\|_{X_T}^2(1+t)^{\gamma/3},
\end{aligned} \tag{3.3}$$

respectively. Next we consider the L^∞-bound for $u(t)$. In the case of $t \le 1$, the standard Sobolev embedding and (3.2) lead to

$$(1+t)\|u(t,\cdot)\|_{L^\infty} \le 2\|u(t,\cdot)\|_{L^\infty} \le C\|u(t,\cdot)\|_{H^{s,0}} \le C(\varepsilon + \|u\|_{X_T}^2). \tag{3.4}$$

From now on, we focus on $t \ge 1$. We set

$$\alpha(t,\xi) = \mathcal{G}\big(\mathcal{U}(-t)u(t,\cdot)\big)(\xi)$$

and

$$r(t,\xi) = \mathcal{G}\big(\mathcal{U}(-t)F(u(t,\cdot))\big)(\xi) - \frac{1}{t}F(\alpha(t,\xi))$$

so that

$$\begin{aligned} i\partial_t\alpha(t,\xi) =& \mathcal{G}\big(\mathcal{U}(-t)(i\partial_t + \Lambda)u(t,\cdot)\big)(\xi) \\ =& \mathcal{G}\big(\mathcal{U}(-t)F(u(t,\cdot))\big)(\xi) \\ =& \frac{1}{t}F(\alpha(t,\xi)) + r(t,\xi). \end{aligned}$$

By the Sobolev inequality and (3.3), we get

$$\|\alpha(1,\cdot)\|_{L^\infty} \le C\,\|\mathcal{U}(-1)u(1,\cdot)\|_{H^{0,s}} \le C\varepsilon + C\|u\|_{X_T}^2.$$

By Lemma 2.4, we have

$$\begin{aligned} \|r(t,\cdot)\|_{L^\infty} &\le Ct^{-1-\gamma}\|\mathcal{U}(-t)u(t)\|_{H^{0,s}}^2 \\ &\le Ct^{-1-\gamma}\Big\{(1+t)^{\frac{\gamma}{3}}\|u\|_{X_T}\Big\}^2 \\ &\le Ct^{-1-\frac{\gamma}{3}}\|u\|_{X_T}^2. \end{aligned} \tag{3.5}$$

Then it follows from the straightforward calculation that

$$\begin{aligned} \partial_t\Big(\nu_A(\alpha(t,\xi))^2\Big) &= 2\,\mathrm{Im}\Big\langle i\partial_t\alpha(t,\xi), A\alpha(t,\xi)\Big\rangle_{\mathbb{C}^3} \\ &= \frac{2}{t}\,\mathrm{Im}\langle F(\alpha(t,\xi)), A\alpha(t,\xi)\rangle_{\mathbb{C}^3} + 2\,\mathrm{Im}\langle r(t,\xi), A\alpha(t,\xi)\rangle_{\mathbb{C}^3} \\ &\le 0 + 2\nu_A(r(t,\xi))\nu_A(\alpha(t,\xi)), \end{aligned} \tag{3.6}$$

which leads to

$$\partial_t\sqrt{\delta + \nu_A(\alpha(t,\xi))^2} \le \nu_A(r(t,\xi)) \le C\|r(t,\cdot)\|_{L^\infty} \le C\|u\|_{X_T}^2 t^{-1-\frac{\gamma}{3}}$$

for any $\delta > 0$. Integrating with respect to t, and letting $\delta \to +0$, we obtain

$$\nu_A(\alpha(t,\xi)) \le C\|\alpha(1,\cdot)\|_{L^\infty} + C\|u\|_{X_T}^2\int_1^t \tau^{-1-\frac{\gamma}{3}}d\tau,$$

whence

$$\|\alpha(t,\cdot)\|_{L^\infty} \le C\varepsilon + C\|u\|_{X_T}^2. \tag{3.7}$$

From Lemma 2.2 and (3.3), we deduce that

$$\begin{aligned}\|u(t,\cdot)\|_{L^\infty} &\le t^{-1}\|\alpha(t,\cdot)\|_{L^\infty} + Ct^{-1-2\gamma/3}(t^{-\gamma/3}\|\mathcal{U}(-t)u(t,\cdot)\|_{H^{0,s}})\\ &\le Ct^{-1}(\varepsilon + \|u\|_{X_T}^2)\end{aligned} \tag{3.8}$$

for $t \ge 1$.

By (3.2), (3.3), (3.4) and (3.8), we obtain the desired estimate. □

4 Proof of Theorem 1.1

Now we are in a position to finish the proof of Theorem 1.1. It is enough to show

$$\sup_{\xi\in\mathbb{R}^2} \nu_A(\alpha(t,\xi)) \le \frac{C}{\log t} \tag{4.1}$$

for $t \ge 2$, because we already know that

$$(1+t)\|u(t,\cdot)\|_{L^\infty} \le C\sup_{\xi\in\mathbb{R}^2} \nu_A(\alpha(t,\xi)) + C\varepsilon t^{-2\gamma/3}$$

by virtue of Lemma 2.2, (3.1) and (3.3).

To prove (4.1), we put $\Phi(t) := \nu_A(\alpha(t,\xi))^2$ (with $\xi \in \mathbb{R}^2$ being regarded as a parameter) and compute

$$\frac{d}{dt}\Big((\log t)^3\Phi(t)\Big) = (\log t)^3\frac{d\Phi}{dt}(t) + \frac{3(\log t)^2}{t}\Phi(t).$$

Similarly to (3.6), it follows from (2.2), (3.5), (3.7) and (3.1) that

$$\begin{aligned}\frac{d\Phi}{dt}(t) &= \frac{2}{t}\operatorname{Im}\langle F(\alpha(t,\xi)), A\alpha(t,\xi)\rangle_{\mathbb{C}^3} + 2\operatorname{Im}\langle r(t,\xi), A\alpha(t,\xi)\rangle_{\mathbb{C}^3}\\ &\le -\frac{2}{t}C_*\,\Phi(t)^{3/2} + \frac{C\varepsilon^3}{t^{1+\gamma/3}},\end{aligned}$$

where C_* is the constant appearing in (2.2). We also have

$$\Phi(t) = \left(\frac{1}{C_*^2(\log t)^2}\right)^{1/3}\Big\{C_*(\log t)\Phi(t)^{3/2}\Big\}^{2/3} \le \frac{1}{3C_*^2(\log t)^2} + \frac{2C_*}{3}(\log t)\Phi(t)^{3/2}$$

by the Young inequality. Piecing them together, we obtain

$$\frac{d}{dt}\Big((\log t)^3\Phi(t)\Big) \le \frac{C}{t} + \frac{C\varepsilon^3(\log t)^3}{t^{1+\gamma/3}}.$$

Integrating with respect to t, we arrive at

$$(\log t)^3\Phi(t) \le C\varepsilon^2 + \int_2^t \left(\frac{C}{\tau} + \frac{C\varepsilon^3(\log\tau)^3}{\tau^{1+\gamma/3}}\right)d\tau \le C\log t,$$

whence $\Phi(t) \le C/(\log t)^2$ for $t \ge 2$, as required. $\square$

Finally, we discuss the optimality of the decay rate $O((t\log t)^{-1})$. For simplicity, let $\varphi(x) = \delta\psi(x)$ with $\psi(\not\equiv 0) \in H^{s,0} \cap H^{0,s}$ and $\delta > 0$ (note that $\varepsilon = \|\varphi\|_{H^{s,0}} + \|\varphi\|_{H^{0,s}} \le C\delta$). Then we can also show that the solution does not decay strictly faster than $t^{-1}(\log t)^{-1}$ as $t \to \infty$ if δ is small enough. Indeed, suppose that

$$\lim_{t\to\infty} t(\log t)\|u(t,\cdot)\|_{L^\infty} = 0$$

holds true. Then, it follows from Lemma 2.2, (3.1) and (3.3) that

$$\begin{aligned}(\log t)\Phi(t)^{1/2} &\le Ct(\log t)t^{-1}\|\alpha(t,\cdot)\|_{L^\infty}\\ &\le Ct(\log t)\left(\|u(t,\cdot)\|_{L^\infty} + C\delta t^{-1-2\gamma/3}\right)\\ &\to 0 \end{aligned}\tag{4.2}$$

as $t \to \infty$. Hence, if δ is sufficiently small, we have

$$C^*(\log t)\Phi(t)^{1/2} \le 1$$

for $t \ge 2$, where C^* is the constant appearing in (2.2). Similarly to the proof of Theorem 1.1, we have

$$\frac{d}{dt}\left((\log t)^2\Phi(t)\right) \ge \frac{2(\log t)}{t}\Phi(t)\left(1 - C^*(\log t)\Phi(t)^{1/2}\right) - \frac{C(\log t)^2\delta^3}{t^{1+\gamma/3}} \ge -\frac{C(\log t)^2\delta^3}{t^{1+\gamma/3}},$$

which yields

$$(\log t)^2\Phi(t) \ge (\log 2)^2\Phi(2) - \int_2^t \frac{C(\log\tau)^2\varepsilon^3}{\tau^{1+\gamma/3}}d\tau \ge C\delta^2 - C'\delta^3 > 0$$

for small δ with some positive constants C and C'. This contradicts (4.2).

A Appendix

For the convenience of the readers, we give an outline of the proof of the four lemmas stated in Section 2.

Proof of Lemma 2.1. For $s \in [0,2)$, we have

$$\left\| (-\Delta)^{s/2}(|f|f) \right\|_{L^2} \le C\|f\|_{L^\infty}\|(-\Delta)^{s/2}f\|_{L^2} \tag{A.1}$$

and

$$\|(-\Delta)^{s/2}(fg)\|_{L^2} \le C(\|f\|_{L^\infty}\|(-\Delta)^{s/2}g\|_{L^2} + \|(-\Delta)^{s/2}f\|_{L^2}\|g\|_{L^\infty}) \tag{A.2}$$

(see, e.g., [3] and [7] for the proof). The desired estimate follows from them immediately. $\square$

Proof of Lemma 2.2. By a simple calculation, we can see that $\mathcal{U}(t)$ is decomposed into the following forms:

$$\mathcal{U}(t) = \mathcal{M}(t)\mathcal{D}(t)\mathcal{G}\mathcal{M}(t) = \mathcal{M}(t)\mathcal{D}(t)\mathcal{W}(t)\mathcal{G}, \quad t \neq 0, \tag{A.3}$$

where $\mathcal{M}(t) = \mathcal{E}_\theta$ with $\theta = |x|^2/(2t)$, $\big(\mathcal{D}(t)\phi\big)(x) = \frac{1}{t}\phi(\frac{x}{t})$, and $\mathcal{W}(t) = \mathcal{G}\mathcal{M}(t)\mathcal{G}^{-1}$. Note that $|e^{i\theta} - 1| \leq C|\theta|^\gamma$ for $\gamma \in [0,1]$ and $(1+|x|)^{2\gamma - s} \in L^2(\mathbb{R}^2)$ if $s > 1 + 2\gamma$. They imply

$$\begin{aligned}
\|\mathcal{G}(\mathcal{M}(t) - 1)\psi\|_{L^\infty} &\leq C\,\|(\mathcal{M}(t)-1)\psi\|_{L^1} \\
&\leq Ct^{-\gamma}\left\||x|^{2\gamma}\psi\right\|_{L^1} \\
&\leq Ct^{-\gamma}\left\|(1+|x|)^{2\gamma-s}\right\|_{L^2}\|(1+|x|)^s\psi\|_{L^2} \\
&\leq Ct^{-\gamma}\,\|\psi\|_{H^{0,s}}\,.
\end{aligned} \tag{A.4}$$

Since we have

$$\begin{aligned}
\|\phi\|_{L^\infty} &= \|\mathcal{M}(t)\mathcal{D}(t)\mathcal{G}\mathcal{M}(t)\mathcal{U}(-t)\phi\|_{L^\infty} \\
&= t^{-1}\|\mathcal{G}\mathcal{M}(t)\mathcal{U}(-t)\phi\|_{L^\infty},
\end{aligned}$$

it follows from (A.4) that

$$\begin{aligned}
\big|\|\phi\|_{L^\infty} - t^{-1}\|\mathcal{G}\mathcal{U}(-t)\phi\|_{L^\infty}\big| &\leq t^{-1}\|\mathcal{G}(\mathcal{M}(t)-1)\mathcal{U}(-t)\phi\|_{L^\infty} \\
&\leq Ct^{-1-\gamma}\|\mathcal{U}(-t)\phi\|_{H^{0,s}}.
\end{aligned}$$

□

Proof of Lemma 2.3. By (1.3), or equivalently (2.1), we have

$$\mathcal{M}(-t)F(\phi) = F(\mathcal{M}(-t)\phi). \tag{A.5}$$

We also note that

$$\mathcal{U}(t)|x|^s\mathcal{U}(-t) = \mathcal{M}(t)\begin{pmatrix} \frac{t^s(-\Delta)^{s/2}}{m_1^s} & 0 & 0 \\ 0 & \frac{t^s(-\Delta)^{s/2}}{m_2^s} & 0 \\ 0 & 0 & \frac{t^s(-\Delta)^{s/2}}{m_3^s} \end{pmatrix}\mathcal{M}(-t).$$

By using (A.1), (A.2) and the above identities, we deduce that

$$\begin{aligned}
\||x|^s\mathcal{U}(-t)F(\phi)\|_{L^2} &\leq Ct^s\left\|(-\Delta)^{s/2}F(\mathcal{M}(-t)\phi)\right\|_{L^2} \\
&\leq Ct^s\,\|\mathcal{M}(-t)\phi\|_{L^\infty}\left\|(-\Delta)^{s/2}(\mathcal{M}(-t)\phi)\right\|_{L^2} \\
&\leq C\|\phi\|_{L^\infty}\,\||x|^s\mathcal{U}(-t)\phi\|_{L^2}\,.
\end{aligned}$$

□

Proof of Lemma 2.4. We put $\beta = \mathcal{G}\mathcal{U}(-t)\phi$. By (A.3) and (A.5), we have

$$\begin{aligned}\mathcal{G}\mathcal{U}(-t)F(\phi) &= \mathcal{W}(t)^{-1}\mathcal{D}(t)^{-1}\mathcal{M}(t)^{-1}F(\mathcal{M}(t)\mathcal{D}(t)\mathcal{W}(t)\beta)\\ &= \mathcal{W}(t)^{-1}\mathcal{D}(t)^{-1}F(\mathcal{D}(t)\mathcal{W}(t)\beta)\\ &= \frac{1}{t}\mathcal{W}(t)^{-1}F(\mathcal{W}(t)\beta)\\ &= \frac{1}{t}F(\mathcal{G}\mathcal{U}(-t)\phi) + R(t),\end{aligned}$$

where

$$\begin{aligned}R(t) &= \frac{1}{t}(R_1(t) + R_2(t)),\\ R_1(t) &= (1 - \mathcal{W}(t))\mathcal{W}(t)^{-1}F(\mathcal{W}(t)\beta),\\ R_2(t) &= F(\mathcal{W}(t)\beta) - F(\beta).\end{aligned}$$

(A.4) yields

$$\|(\mathcal{W}(t) - 1)\psi\|_{L^\infty} \leq Ct^{-\gamma}\left\|\mathcal{G}^{-1}\psi\right\|_{H^{0,s}} \leq Ct^{-\gamma}\|\psi\|_{H^{s,0}}.$$

Hence, by Lemma 2.1 and the Sobolev embedding, we have

$$\begin{aligned}\|R_1(t)\|_{L^\infty} &\leq Ct^{-\gamma}\left\|\mathcal{W}(t)^{-1}F(\mathcal{W}(t)\beta)\right\|_{H^{s,0}}\\ &\leq Ct^{-\gamma}\|F(\mathcal{W}(t)\beta)\|_{H^{s,0}}\\ &\leq Ct^{-\gamma}\|\mathcal{W}(t)\beta\|_{H^{s,0}}^2\\ &\leq Ct^{-\gamma}\|\beta\|_{H^{s,0}}^2\end{aligned}$$

and

$$\begin{aligned}\|R_2(t)\|_{L^\infty} &\leq C(\|\mathcal{W}(t)\beta\|_{L^\infty} + \|\beta\|_{L^\infty})\|(\mathcal{W}(t) - 1)\beta\|_{L^\infty}\\ &\leq C(\|\mathcal{W}(t)\beta\|_{H^{s,0}} + \|\beta\|_{H^{s,0}}) \cdot Ct^{-\gamma}\|\beta\|_{H^{s,0}}\\ &\leq Ct^{-\gamma}\|\beta\|_{H^{s,0}}^2.\end{aligned}$$

Summing up, we arrive at

$$\|R(t)\|_{L^\infty} \leq Ct^{-1-\gamma}\|\beta\|_{H^{s,0}}^2 = Ct^{-1-\gamma}\|\mathcal{U}(-t)\phi\|_{H^{0,s}}^2,$$

as required. □

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